POWER SUMS OF HECKE EIGENVALUES AND APPLICATION

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ABSTRACT. We sharpen some estimates of Rankin on power sums of Hecke eigenvalues, by using Kim & Shahidi's recent results on higher order symmetric powers. As an application, we improve Kohnen, Lau & Shparlinski's lower bound for the number of Hecke eigenvalues of same signs.

1. INTRODUCTION

Let $k \geqslant 2$ be an even ineteger and $N \geqslant 1$ be squarefree. Denote by $\mathrm{H}_k^*(N)$ the set of all normalized Hecke primitive eigencuspforms of weight k for the congruence modular group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \,(\mathrm{mod}\, N) \right\}.$$

Here the normalization is taken to have $\lambda_f(1) = 1$ in the Fourier series of $f \in \mathrm{H}_k^*(N)$ at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m z > 0). \tag{1.1}$$

Inherited from the Hecke operators, the normalized Fourier coefficient $\lambda_f(n)$ satisfies the following relation

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1.2}$$

for all integers $m \geqslant 1$ and $n \geqslant 1$. In particular, $\lambda_f(n)$ is multiplicative.

Following Deligne [3], for any prime number p there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p) p^{-1/2}, \ \beta_f(p) = 0 & \text{if } p \mid N \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N \end{cases} \tag{1.3}$$

and

$$\lambda_f(p^\nu) = \frac{\alpha_f(p)^{\nu+1} - \beta_f(p)^{\nu+1}}{\alpha_f(p) - \beta_f(p)} \tag{1.4}$$

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for all integers $\nu \geqslant 1$, where $\varepsilon_f(p) = \pm 1$. Hence $\lambda_f(n)$ is real and verifies Deligne's inequality

$$|\lambda_f(n)| \leqslant d(n) \tag{1.5}$$

for all integers $n \geqslant 1$, where $d(n)$ is the divisor function. In particular for each prime number $p \nmid N$ there is $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p). \tag{1.6}$$

See e.g. [7] for basic analytic facts about modular forms.

Positive real moments of Hecke eigenvalues were firstly studied by Rankin ([14], [15]). For $f \in \mathrm{H}_k^*(N)$ and $r \geqslant 0$, consider the sum of the $2r$th power of $|\lambda_f(n)|$:

$$S_f^*(x; r) := \sum_{n \leqslant x} |\lambda_f(n)|^{2r}. \tag{1.7}$$

The method of Rankin [15] illustrates how to obtain optimally the lower and upper bounds for $S_f^*(x; r)$ if we only know that the associated Dirichlet series

$$F_r(s) := \sum_{n \geqslant 1} |\lambda_f(n)|^{2r} n^{-s} \quad (\Re e\, s > 1) \tag{1.8}$$

is invertible for $\Re e\, s \geqslant 1$ (i.e. holomorphic and nonzero for $\Re e\, s \geqslant 1$) when $r = 1, 2$. (The invertibility of these two cases are known by Moreno & Shahidi [13].) Rankin's result ([15], Theorem 1) reads that

$$x(\log x)^{\delta_r^{\mp}} \ll S_f^*(x; r) \ll x(\log x)^{\delta_r^{\pm}} \quad (r \in \mathcal{R}^{\mp}) \tag{1.9}$$

for $x \geqslant x_0(f, r)$, where

$$\mathcal{R}^- := [0, 1] \cup [2, \infty), \qquad \mathcal{R}^+ := [1, 2],$$

and

$$\delta_r^- := 2^{r-1} - 1, \qquad \delta_r^+ := \frac{2^{r-1}}{5}(2^r + 3^{2-r}) - 1.$$

The implied constants in (1.9) depend on f and r.

On the other hand, if the Sato-Tate conjecture holds for newform f, then

$$S_f^*(x; r) \sim C_r(f) x (\log x)^{\theta_r} \quad (x \to \infty), \tag{1.10}$$

where $C_r(f)$ is a positive constant depending on f, r and

$$\theta_r := \frac{4^r \Gamma(r + \frac{1}{2})}{\sqrt{\pi}\Gamma(r + 2)} - 1.$$

Very recently, Tenenbaum [20] improved Rankin's exponent $\delta_{1/2}^+ = 0.0651\cdots$ to $\rho_{1/2}^+ = 0.1185\cdots$ (see (1.13) below for the definition of ρ_r^+), as an application of his general result on the mean values of multiplicative functions and the fact that $F_3(s)$ and $F_4(s)$ are invertible for $\Re e\, s \geqslant 1$, proven in the excellent work of Kim & Shahidi [9]. Although the result ([20], Corollary) is stated only for Ramanujan's τ-function, it is apparent that Tenenbaum's method applies to establish the upper bound for $S_f^*(x; r)$ in (1.11) below. It should be pointed out that Tenenbaum's approach is different from that of Rankin and does not give a lower bound for $S_f^*(x; r)$.

The first aim of this paper is to improve the lower and upper bounds in (1.9), by generalizing Rankin's method to incorporate the aforementioned results of Kim & Shahidi on $F_3(s)$ and $F_4(s)$.

Theorem 1. *For any $f \in \mathrm{H}_k^*(N)$, we have*

$$x(\log x)^{\rho_r^{\mp}} \ll S_f^*(x;r) \ll x(\log x)^{\rho_r^{\pm}} \quad (r \in \mathscr{R}^{\mp}) \tag{1.11}$$

for $x \geqslant x_0(f,r)$, where

$$\mathscr{R}^- := [0,1] \cup [2,3] \cup [4,\infty), \qquad \mathscr{R}^+ := [1,2] \cup [3,4], \tag{1.12}$$

and

$$\begin{cases} \rho_r^- := \dfrac{3^{r-1}-1}{2}, \\ \rho_r^+ := \dfrac{102+7\sqrt{21}}{210}\Big(\dfrac{6-\sqrt{21}}{5}\Big)^r + \dfrac{102-7\sqrt{21}}{210}\Big(\dfrac{6+\sqrt{21}}{5}\Big)^r + \dfrac{4^r}{35} - 1. \end{cases} \tag{1.13}$$

The implied constants in (1.11) depend on f and r.

The upper bound part in (1.11) are essentially due to Tenenbaum [20], since his method with a minuscule modification allows to obtain this result. The lower bound part is new. The following table illustrates progress against Rankin's (1.9) and the difference from the conjectured values (1.10).

r	0	0.5	1	1.5	2	2.5	3	3.5	4
δ_r^-	-0.5	-0.2929	0	0.4142	1	1.8284	3	4.6569	7
ρ_r^-	-0.3333	-0.2113	0	0.3660	1	2.0981	4	7.2945	13
θ_r	0	-0.1512	0	0.3581	1	2.1043	4	7.2781	13
ρ_r^+	0	-0.1185	0	0.3502	1	2.1112	4	7.2576	13
δ_r^+	0	-0.0652	0	0.2899	1	2.5266	5.6667	12.0177	24.7778

In order to detect sign changes or cancellations among $\lambda_f(n)$, it is natural to study summatory function

$$S_f(x) := \sum_{n \leqslant x} \lambda_f(n) \tag{1.14}$$

and compare it with (1.11). There is a long history on the investigation of the upper estimate for $S_f(x)$. In 1927, Hecke [6] showed

$$S_f(x) \ll_f x^{1/2}$$

for all $f \in \mathrm{H}_k^*(N)$ and $x \geqslant 1$. Subsequent improvements came with the use of the identity:

$$\frac{1}{\Gamma(r+1)} \sum_{n \leqslant x} (x-n)^r a_f(n) = \frac{1}{(2\pi)^3} \sum_{n \geqslant 1} \Big(\frac{x}{n}\Big)^{(k+3)/2} a_f(n) J_{k+3}\big(4\pi\sqrt{nx}\big),$$

where $a_f(n) := \lambda_f(n)n^{(k-1)/2}$ and $J_k(t)$ is the first kind Bessel functions. Such an identity was first given by Wilton [22] in which only the case of Ramanujan's τ-function was stated, and later generalized by Walfisz [21] to other forms. Let ϑ be the constant satisfying

$$|\lambda_f(n)| \ll n^{\vartheta} \quad (n \geqslant 1).$$

Walfisz proved that

$$S_f(x) \ll_f x^{(1+\vartheta)/3} \quad (x \geqslant 1). \tag{1.15}$$

Inserting the values of ϑ in the historical record into (1.15) yields

$$S_f(x) \ll_{f,\varepsilon} \begin{cases} x^{11/24+\varepsilon} & \text{Kloosterman [10]} \\ x^{4/9+\varepsilon} & \text{Davenport [1], Salié [17]} \\ x^{5/12+\varepsilon} & \text{Weil [23]} \\ x^{1/3+\varepsilon} & \text{Deligne [3]} \end{cases}$$

for any $\varepsilon > 0$. Hafner & Ivić ([5], Theorem 1) removed the factor x^{ε} of Deligne's result. On the other hand, by combining Walfisz' method with his idea in the study of (1.7), Rankin [16] showed that

$$S_f(x) \ll_{f,\varepsilon} x^{1/3}(\log x)^{\delta_{1/2}^{+}+\varepsilon} \tag{1.16}$$

for any $\varepsilon > 0$ and $x \geqslant 2$.

Here we propose a better bound, by combining Walfisz' method [21] and Tenenbaum's approach [20]. It is worthy to point out that Tenenbaum's method is not only to improve $\delta_{1/2}^{+}$ to $\rho_{1/2}^{+}$ but also remove the ε in (1.16).

Theorem 2. *For $f \in \mathrm{H}_k^*(N)$, we have*

$$S_f(x) \ll x^{1/3}(\log x)^{\rho_{1/2}^{+}} \tag{1.17}$$

for $x \geqslant 2$, where the implied constant depends on f.

In the opposite direction, Hafner & Ivić ([5], Theorem 2) proved that there is a positive constant D such that

$$S_f(x) = \Omega_{\pm}\left(x^{1/4}\exp\left\{\frac{D(\log_2 x)^{1/4}}{(\log_3 x)^{3/4}}\right\}\right),$$

where $\log_r$ denotes the r-fold iterated logarithm.

As an application of Theorems 1 and 2, we consider the quantities

$$\mathscr{N}_f^{\pm}(x) := \sum_{\substack{n\leqslant x \\ \lambda_f(n)\gtrless 0}} 1. \tag{1.18}$$

Very recently Kohnen, Lau & Shparlinski ([11], Theorem 1) proved

$$\mathscr{N}_f^{\pm}(x) \gg_f \frac{x}{(\log x)^{17}} \tag{1.19}$$

for $x \geqslant x_0(f)$.†

Here we propose a better bound.

†It is worthy to indicate that they gave explicit values for the implied constant in $\ll$ and $x_0(f)$.

Corollary 1. *For any $f \in \mathrm{H}_k^*(N)$, we have*

$$\mathscr{N}_f^{\pm}(x) \gg \frac{x}{(\log x)^{1-1/\sqrt{3}}}$$

for $x \geqslant x_0(f)$, where the implied constant depends on f. If we assume Sato-Tate's conjecture, the exponent $1 - 1/\sqrt{3} \approx 0.422$ can be improved to $2 - 16/(3\pi) \approx 0.302$.

In a joint paper with Lau [12], we shall remove the logarithmic factor by a completely different method.

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2. Method of Rankin

Let $k \geqslant 2$ be an even integer, $N \geqslant 1$ be squarefree, $f \in \mathrm{H}_k^*(N)$ and $r > 0$. Following Rankin's idea [15], we shall find two optimal multiplicative functions $\lambda_{f,r}^{\pm}(n)$ such that

$$\lambda_{f,r}^{\mp}(p^\nu) \leqslant |\lambda_f(p^\nu)|^{2r} \leqslant \lambda_{f,r}^{\pm}(p^\nu) \quad (r \in \mathscr{R}^{\mp}) \tag{2.1}$$

for all primes p and integers $\nu \geqslant 1$, and furthermore, their associated Dirichlet series $\Lambda_{f,r}^{\pm}(s)$ (see (2.8) below) in the half-plane $\Re e\, s \geqslant 1$ is controlled by $F_j(s)$ for $j = 1, \dots, 4$. Then we can apply Tauberian theorems to obtain the asymptotic behaviour of the summatory functions of $\lambda_{f,r}^{\pm}(n)$.

2.1. Construction of $\lambda_{f,r}^{+}(n)$.

For $\boldsymbol{a} := (a_1, \dots, a_4) \in \mathbb{R}^4$ and $r > 0$, consider the function

$$h_r(t; \boldsymbol{a}) := t^r - a_1 t - a_2 t^2 - a_3 t^3 - a_4 t^4 \qquad (0 \leqslant t \leqslant 1) \tag{2.2}$$

and let

$$\kappa_- := \tfrac{1}{4}, \qquad \eta_- := \tfrac{3}{4}, \qquad \kappa_+ := \tfrac{6-\sqrt{21}}{20}, \qquad \eta_+ := \tfrac{6+\sqrt{21}}{20}. \tag{2.3}$$

In Subsection 2.3, we shall explain the reason behind this choice.

Lemma 2.1. *If the function $h_r(t; \boldsymbol{a})$ defined by (2.2) satisfies*

$$h_r'(\kappa_-; \boldsymbol{a}) = h_r'(\eta_-; \boldsymbol{a}) = h_r(\kappa_-; \boldsymbol{a}) = h_r(\eta_-; \boldsymbol{a}) = 0,$$

then

$$a_j = a_j^- := \frac{P_j^-(\kappa_-, \eta_-) - P_j^-(\eta_-, \kappa_-)}{(\kappa_- - \eta_-)^3} \tag{2.4}$$

for $1 \leqslant j \leqslant 4$, where

$$\begin{aligned}
P_1^-(\kappa, \eta) &:= \{(4 - r)\kappa + (r - 2)\eta\}\kappa^{r-1}\eta^2, \\
P_2^-(\kappa, \eta) &:= \{(2r - 8)\kappa^2 + (1 - r)\kappa\eta + (1 - r)\eta^2\}\kappa^{r-2}\eta, \\
P_3^-(\kappa, \eta) &:= \{(4 - r)\kappa^2 + (4 - r)\kappa\eta + 2(r - 1)\eta^2\}\kappa^{r-2}, \\
P_4^-(\kappa, \eta) &:= \{(r - 3)\kappa + (1 - r)\eta\}\kappa^{r-2}.
\end{aligned}$$

Proof. This can be done by routine calculation. $\square$

Lemma 2.2. *If the function $h_r(t;\boldsymbol{a})$ defined by (2.2) is such that*

$$\begin{cases} h_r'(\kappa_+;\boldsymbol{a}) = h_r'(\eta_+;\boldsymbol{a}) = 0, \\ h_r(\kappa_+;\boldsymbol{a}) = h_r(\eta_+;\boldsymbol{a}) = h_r(1;\boldsymbol{a}), \end{cases}$$

then

$$a_j = a_j^+ := \frac{P_j^+(\kappa_+,\eta_+) - P_j^+(\eta_+,\kappa_+)}{(\kappa_+-1)^2(\eta_+-1)^2(\kappa_+-\eta_+)^3} \tag{2.5}$$

for $1 \leqslant j \leqslant 4$, where

$$\begin{aligned} P_1^+(\kappa,\eta) &:= r\kappa^{r-1}\eta(\kappa-1)(\eta-\kappa)(\kappa\eta+2\kappa+\eta)(\eta-1)^2 \\ &\quad + 2(\kappa^r-1)\kappa\eta(\eta-1)^2(2\kappa\eta+4\kappa-\eta^2-2\eta-3), \\ P_2^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2(2\kappa\eta+\kappa+\eta^2+2\eta) \\ &\quad + (\eta^r-1)(\kappa-1)^2(8\kappa\eta^2+4\eta^2-\eta\kappa^2-2\kappa\eta-3\eta-\kappa^3-2\kappa^2-3\kappa), \\ P_3^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa+2\eta+1)(\eta-\kappa)(\eta-1)^2 \\ &\quad + 2(\kappa^r-1)(2\kappa^2+2\kappa\eta-\eta^2-2\eta-1)(\eta-1)^2, \\ P_4^+(\kappa,\eta) &:= r\kappa^{r-1}(\kappa-1)(\kappa-\eta)(\eta-1)^2 + (\eta^r-1)(\kappa-1)^2(3\eta-\kappa-2). \end{aligned}$$

Proof. This is done by routine calculation as well. $\square$

Lemma 2.3. *Let $\boldsymbol{a}^\pm := (a_1^\pm, \dots, a_4^\pm)$, where each $a_i^\pm$ is given by the value in Lemmas 2.1-2.2, respectively. Then for $0 \leqslant t \leqslant 1$ we have*

$$h_r(t;\boldsymbol{a}^-) \gtreqless 0 \quad \textit{and} \quad h_r(t;\boldsymbol{a}^+) \lesseqgtr h_r(1;\boldsymbol{a}^+) \quad \textit{for} \quad r \in \mathscr{R}^\mp.$$

Proof. We have

$$h_r^{(4)}(t;\boldsymbol{a}^-) = r(r-1)(r-2)(r-3)t^{r-4} - 24a_4^-,$$

so $h_r^{(4)}(t;\boldsymbol{a}^-)$ has at most one zero for $t>0$ and $h_r^{(i)}(t;\boldsymbol{a}^-)$ has at most $5-i$ zeros for $t>0$ ($i=3,2,1,0$). Since $h_r(\kappa_-;\boldsymbol{a}^-) = h_r(\eta_-;\boldsymbol{a}^-) = h_r(0;\boldsymbol{a}^-)$, it follows that $h_r'(\xi_-;\boldsymbol{a}^-) = h_r'(\xi_-';\boldsymbol{a}^-) = 0$ for some $\xi_- \in (0,\kappa_-)$ and $\xi_-' \in (\kappa_-,\eta_-)$. Therefore ξ_-, κ_-, ξ_-' and η_- are the only zeros of $h_r'(t;\boldsymbol{a}^-)$ in $(0,1)$.

Now

$$h_r''(\kappa_-;\boldsymbol{a}^-) = 8\cdot 4^{-r}(2r^2-2r+3+2r3^{r-2}-11\cdot 3^{r-2})$$

and

$$h_r''(\eta_-;\boldsymbol{a}^-) = 8\cdot 4^{-r}(2r^2-6r-3-2r3^r+43\cdot 3^{r-2}).$$

From these, it is easy to verify that

$$h_r''(\kappa_-;\boldsymbol{a}^-),\ h_r''(\eta_-;\boldsymbol{a}^-) \begin{cases} \gtreqless 0 & \text{if } r \in \mathring{\mathscr{R}}^\mp, \\ = 0 & \text{if } r = 1,2,3,4, \end{cases}$$

where $\mathring{\mathscr{R}}^\mp$ denotes the interior of $\mathscr{R}^\mp$. Hence $h_r(t;\boldsymbol{a}^-)$ takes its mimimum (maximum, respectively) values in $[0,1]$ at 0, κ_-, η_- when $r \in \mathring{\mathscr{R}}^-$ ($r \in \mathring{\mathscr{R}}^+$, respectively). Moreover, $h_r(t;\boldsymbol{a}^-)$ has local maxima (minima, respectively) at ξ_-, ξ_-' when $r \in \mathring{\mathscr{R}}^-$ ($r \in \mathring{\mathscr{R}}^+$, respectively). This proves the assertion about $h_r(t;\boldsymbol{a}^-)$.

Similarly we can prove the corresponding result on $h_r(t;\boldsymbol{a}^+)$. $\square$

Now we define the multiplicative function $\lambda_{f,r}^{\pm}(n)$ by

$$\lambda_{f,r}^{\mp}(p^\nu) := \begin{cases} \sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^{\mp}\lambda_f(p)^{2j} & \text{if } \nu = 1 \text{ and } r > 0, \\ 0 & \text{if } \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\mp}, \\ |\lambda_f(p^\nu)|^{2r} & \text{if } \nu \geqslant 2 \text{ and } r \in \mathscr{R}^{\pm}, \end{cases} \tag{2.6}$$

where

$$a_0^- := 0 \quad \text{and} \quad a_0^+ := 1 - a_1^+ - a_2^+ - a_3^+ - a_4^+. \tag{2.7}$$

In view of (1.6), we can apply Lemma 2.3 with $t = |\cos\theta_f(p)|$ to deduce that the inequality (2.1) hold for all primes p and integers $\nu \geqslant 1$. Thanking to the multiplicativity, these inequalities also hold for all integers $n \geqslant 1$.

2.2. **Dirichlet series associated to $\lambda_{f,r}^{\pm}(n)$.** For $f \in \mathrm{H}_k^*(N)$, $r > 0$ and $\Re e\, s > 1$, we define

$$\Lambda_{f,r}^{\pm}(s) := \sum_{n\geqslant 1} \lambda_{f,r}^{\pm}(n) n^{-s}. \tag{2.8}$$

Next we shall study their analytic properties in the half-plane $\Re e\, s \geqslant 1$ by using the higher order symmetric power L-functions $L(s, \mathrm{sym}^m f)$ associated to $f \in \mathrm{H}_k^*(N)$, due to Gelbart & Jacquet [4] for $m = 2$, Kim & Shahidi ([8], [9]) for $m = 3, 4, 5, 6, 7, 8$. Here the symmetric mth power associated to f is defined as

$$L(s, \mathrm{sym}^m f) := \prod_p \prod_{0\leqslant j\leqslant m} \left(1 - \alpha_f(p)^{m-j}\beta_f(p)^j p^{-s}\right)^{-1}$$

for $\Re e\, s > 1$, where $\alpha_f(p)$ and $\beta_f(p)$ are given by (1.3) and (1.4). According to the literature mentioned above, it is known that the function $L(s, \mathrm{sym}^m f)$ for $m = 2, 3, \dots, 8$ is invertible for $\Re e\, s \geqslant 1$.

We start to study $F_1(s)$, $F_2(s)$, $F_3(s)$ and $F_4(s)$.

Lemma 2.4. *Let $k \geqslant 2$ be an even integer, $N \geqslant 1$ be squarefree and $f \in \mathrm{H}_k^*(N)$. For $j = 1, 2, 3, 4$ and $\Re e\, s > 1$, we have*

$$F_j(s) = \zeta(s)^{m_j} G_j(s) H_j(s), \tag{2.9}$$

where

$$m_1 := 1, \qquad m_2 := 2, \qquad m_3 := 5, \qquad m_4 := 14, \tag{2.10}$$

and

$$\begin{aligned} G_1(s) &:= L(s, \mathrm{sym}^2 f), \\ G_2(s) &:= L(s, \mathrm{sym}^2 f)^3 L(s, \mathrm{sym}^4 f), \\ G_3(s) &:= L(s, \mathrm{sym}^2 f)^9 L(s, \mathrm{sym}^4 f)^5 L(s, \mathrm{sym}^6 f), \\ G_4(s) &:= L(s, \mathrm{sym}^2 f)^{34} L(s, \mathrm{sym}^4 f)^{20} L(s, \mathrm{sym}^6 f)^7 L(s, \mathrm{sym}^8 f) \end{aligned}$$

are invertible for $\Re e\, s \geqslant 1$. Here the function $H_j(s)$ admits a Dirichlet series convergent absolutely in $\Re e\, s > \frac{1}{2}$ and $H_j(s) \neq 0$ for $\Re e\, s = 1$.

Proof. Write x for the trace of a local factor of $L(s,f)$ (i.e. $\alpha_f(p)+\beta_f(p)$), and denote by $T_n(x)$ the polynomial for the trace of its symmetric nth power. Then

$$\begin{aligned}T_2 &= x^2-1,\\ T_4 &= x^4-3x^2+1,\\ T_6 &= x^6-5x^4+6x^2-1,\\ T_8 &= x^8-7x^6+15x^4-10x^2+1,\end{aligned}$$

from which we deduce

$$\begin{aligned}x^2 &= 1+T_2,\\ x^4 &= 2+3T_2+T_4,\\ x^6 &= 5+9T_2+5T_4+T_6,\\ x^8 &= 14+34T_2+20T_4+7T_6+T_8.\end{aligned}$$

This implies (2.9). By using results on $L(s,\mathrm{sym}^m f)$ mentioned above, $G_j(s)$ is invertible for $\Re e\, s \geqslant 1$. This completes the proof. □

Lemma 2.5. *Let $k\geqslant 2$ be an even integer, $N\geqslant 1$ be squarefree and $f\in \mathrm{H}_k^*(N)$. For $r>0$ and $\Re e\, s>1$, we have*

$$\Lambda_{f,r}^{\pm}(s) = \zeta(s)^{\rho_r^{\pm}+1} H_{f,r}^{\pm}(s), \tag{2.11}$$

where

$$\rho_r^{\pm} := 2^{2r-8}(2^8 a_0^{\pm} + 2^6 a_1^{\pm} + 2^4\cdot 2a_2^{\pm} + 2^2\cdot 5a_3^{\pm} + 14a_4^{\pm}) - 1 \tag{2.12}$$

and $H_{f,r}^{\pm}(s)$ is invertible for $\Re e\, s\geqslant 1$.

Proof. By definition (2.6), for $\Re e\, s>1$ we can write

$$\begin{aligned}\Lambda_{f,r}^{-}(s) &= \prod_p \Big(1+\sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^- \lambda_f(p)^{2j} p^{-s}\Big)\\ &= \prod_{0\leqslant j\leqslant 4} F_j(s)^{2^{2(r-j)}a_j^-} H_r^-(s)\end{aligned}$$

for $r\in\mathscr{R}^-$, and

$$\begin{aligned}\Lambda_{f,r}^{-}(s) &= \prod_p \Big(1+\sum_{0\leqslant j\leqslant 4} 2^{2(r-j)} a_j^- \lambda_f(p)^{2j} p^{-s} + \sum_{\nu\geqslant 2} |\lambda_f(p^\nu)|^{2r} p^{-\nu s}\Big)\\ &= \prod_{0\leqslant j\leqslant 4} F_j(s)^{2^{2(r-j)}a_j^-} H_r^-(s)\end{aligned}$$

for $r\in\mathscr{R}^+$, where $F_0(s)=\zeta(s)$ is the Riemann zeta-function and $H_r^-(s)$ is a Dirichlet series absolutely convergent for $\Re e\, s>\frac12$ such that $H_r^-(s)\neq 0$ for $\Re e\, s=1$. Now the desired result with the sign '$-$' follows from Lemma 2.4. The other part can be treated in the same way. □

2.3. **Optimalisation of $\lambda_{f,r}^{\pm}(p)$ and choice of $\kappa_\pm,\eta_\pm$.** If we regard $\kappa_\pm,\eta_\pm$ as parameters, the $\rho_r^{\pm}$ given by (2.12) are functions of these parameters. We choose $(\kappa_\pm,\eta_\pm)$ in $(0,1)^2$ optimally, which can be done by using formal calculation via Maple. Their values are given by (2.3).

3. Proof of Theorem 1

In view of Lemma 2.5 and the classical fact on $\zeta(s)$, we can write

$$\Lambda_{f,r}^{\pm}(s) = \frac{H_{f,r}^{\pm}(1)}{(s-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(s) \tag{3.1}$$

in some neighbourhood of $s = 1$ with $\Re e\, s > 1$, where $H_{f,r}^{\pm}(1) \neq 1$ and $g_{f,r}^{\pm}(s)$ is holomorphic at $s = 1$. Since $\lambda_{f,r}^{\pm}(n) \geqslant 0$, we can apply Delange's tauberian theorem [2] to write

$$\sum_{n \leqslant x} \lambda_{f,r}^{\pm}(n) \sim H_{f,r}^{\pm}(1) x (\log x)^{\rho_r^{\pm}} \quad (x \to \infty). \tag{3.2}$$

Now Theorem 1 follows from (2.1) and (3.2).

4. Proof of Theorem 2

By (3.1), it follows that

$$\prod_p \left(1 + \sum_{\nu \geqslant 1} \frac{\lambda_{f,r}^{\pm}(p^{\nu})}{p^{\nu\sigma}}\right) = \frac{H_{f,r}^{\pm}(1)}{(\sigma-1)^{\rho_r^{\pm}+1}} + g_{f,r}^{\pm}(\sigma)$$

for $\sigma > 1$. From this, (2.6), (2.7) and Deligne's inequality, we deduce that

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p)}{p^{\sigma}} = (\rho_r^{\pm} + 1)\log(\sigma-1)^{-1} + C_{f,r}^{\pm} + o(1) \quad (\sigma \to 1+),$$

where $C_{f,r}^{\pm}$ is some constant.

On the other hand, the prime number theorem implies, by a partial integration, that

$$\sum_p p^{-\sigma} = \log(\sigma-1)^{-1} + C + o(1) \quad (\sigma \to 1+),$$

where C is an absolute constant. Thus the preceding relation can be written as

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm}+1)}{p^{\sigma}} = C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C + o(1) \quad (\sigma \to 1+). \tag{4.1}$$

According to Exercise II.7.8 of [19], the formula (4.1) implies

$$\sum_p \frac{\lambda_{f,r}^{\pm}(p) - (\rho_r^{\pm}+1)}{p} = C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C.$$

Hence

$$\sum_{p \leqslant x} \frac{\lambda_{f,r}^{\pm}(p)}{p} = (\rho_r^{\pm}+1)\log_2 x + C_{f,r}^{\pm} + (\rho_r^{\pm}+1)C + o(1) \quad (x \to \infty).$$

Now we apply a well known result of Shiu [18] and (2.1) to write

$$
\begin{aligned}
\sum_{x\leqslant n\leqslant x+z} |\lambda_f(n)|^{2r} &\ll \frac{z}{\log x}\exp\biggl(\sum_{p\leqslant x}\frac{|\lambda_f(p)|^{2r}}{p}\biggr)\\
&\ll \frac{z}{\log x}\exp\biggl(\sum_{p\leqslant x}\frac{\lambda_{f,r}^{+}(p)}{p}\biggr)\\
&\ll z(\log x)^{\rho_r^+}
\end{aligned}
\tag{4.2}
$$

for $r \in \mathscr{R}^-$, any $\varepsilon > 0$, $x \geqslant x_0(\varepsilon)$ and $x^{1/4} \leqslant z \leqslant x$. Using this with $r = \frac{1}{2}$ in (9) of [16], the first term on the right-hand side of (10) of [16] is replaced by $x^{1/2}z^{-1/2}(\log x)^{\rho_{1/2}^+}$. Applying (4.2) with $r = \frac{1}{2}$ again to the second term on the right-hand side of (10) of [16], it follows that

$$S_f(x) \ll x^{1/2}z^{-1/2}(\log x)^{\rho_{1/2}^+} + z(\log x)^{\rho_{1/2}^+}.$$

Taking $z = x^{1/3}$, we obtain the required result when the level is $N = 1$. The general case can be treated similarly as indicated in [16]. □

5. Proof of Corollary 1

By comparing (1.17) and the lower bound part in (1.11) with $r = \frac{1}{2}$, it is easy to deduce that

$$\sum_{\substack{n\leqslant x\\ \lambda_f(n)\gtrless 0}} |\lambda_f(n)| \gg_f x(\log x)^{\rho_{1/2}^-}$$

for $x \geqslant x_0(f)$. Since $\rho_{1/2}^- = -(1 - 1/\sqrt{3})/2$ and $\rho_1^+ = 0$, a simple application of the Cauchy-Schwarz inequality yields the following result.

The second assertion can be obtained by noticing that $\theta_{1/2} = 8/(3\pi) - 1$. □

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